

On the noncommutative deformation of the operator graph corresponding to the Klein group

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Abstract

We study the noncommutative operator graph \mathcal{L}_θ depending on complex parameter θ recently introduced by M.E. Shirokov to construct channels with positive quantum zero-error capacity having vanishing n-shot capacity. We define the noncommutative group G and the algebra \mathcal{A}_θ which is a quotient of $\mathbb{C}G$ with respect to the special algebraic relation depending on θ such that the matrix representation ϕ of \mathcal{A}_θ results in the algebra \mathcal{M}_θ generated by \mathcal{L}_θ . In the case of $\theta = \pm 1$ ϕ is degenerated to the faithful representation of $\mathbb{C}K_4$, where K_4 is the Klein group. Thus, \mathcal{L}_θ can be considered as a noncommutative deformation of the graph associated with the Klein group.

1 Introduction.

Denote $\mathfrak{B}(\mathcal{H})$, $\mathfrak{T}(\mathcal{H})$ and $\mathfrak{S}(\mathcal{H})$ the algebra of all bounded operators, the Banach space of all trace class operators and the convex set of positive unit-trace operators (quantum states) in a Hilbert space \mathcal{H} , respectively. A completely positive trace-preserving linear map $\Phi : \mathfrak{T}(\mathcal{H}) \rightarrow \mathfrak{T}(\mathcal{H})$ is said to be a quantum channel. Given a quantum channel Φ one can pick up its Kraus decomposition of the form

$$\Phi(\rho) = \sum_n V_n \rho V_n^*, \quad (1)$$

where $V_n \in \mathfrak{B}(\mathcal{H})$, $\sum_n V_n^* V_n = I$, I is the identity operator in \mathcal{H} , $\rho \in \mathfrak{S}(\mathcal{H})$ [1]. Throughout all this paper the Hilbert space \mathcal{H} is supposed to be finite-dimensional.

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In [2] the noncommutative graph $\mathcal{G}(\Phi)$ of a quantum channel Φ was introduced. Let us consider the Kraus decomposition (1), then $\mathcal{G}(\Phi)$ is the operator subspace

$$\mathcal{G}(\Phi) = \overline{\text{Lin}(V_j^* V_k)}. \quad (2)$$

In [3, 4] it was shown that the operator subspace \mathcal{S} is associated with some channel in the sense of (2) iff $I \in \mathcal{S}$ and $\mathcal{S}^* = \mathcal{S}$. It seems to be interesting to study the algebraic structure of (2). For example, the question arises whether it is possible to show that $\mathcal{G}(\Phi)$ is an image of the representation of the algebra $\mathbb{C}G$ associated with some noncommutative group G ? Here we shall consider this question for an important particular case.

Consider the operator subspace $\mathcal{L}_\theta \subset \text{Mat}_4(\mathbb{C})$:

$$\begin{pmatrix} a & b & c\theta & d \\ b & a & d & c/\theta \\ c/\theta & d & a & b \\ d & c\theta & b & a \end{pmatrix} \quad (3)$$

for $\theta \in \mathbb{C}^*$. The operator graph (3) was introduced in [5] to construct channels with positive quantum zero-error capacity having vanishing n-shot capacity.

A finite-dimensional channel $\Phi : \mathfrak{S}(\mathcal{H}) \rightarrow \mathfrak{S}(\mathcal{H})$ is called pseudo-diagonal [6] if

$$\Phi(\rho) = \sum_{i,j} c_{ij} \langle \psi_i | \rho | \psi_j \rangle |i\rangle \langle j| \quad (4)$$

where $\{c_{ij}\}$ is a Gram matrix of a collection of unit vectors, $\{|\psi_i\rangle\}$ is a collection of vectors in \mathcal{H} such that $\sum_i |\psi_i\rangle \langle \psi_i| = I$ and $\{|i\rangle\}$ is an orthonormal basis in \mathcal{H} . Pseudo-diagonal channels are complementary to entanglement-breaking channels and vice versa. In [5] it was shown that the noncommutative graph \mathcal{L}_θ can be associated with a family of pseudo-diagonal channels (4) depending on the parameter θ .

In the present paper we shall show that \mathcal{L}_θ can be considered as a image of representation π_θ of the ring generated by the group G with three generators x, y, z satisfying the relations

$$x^2 = y^2 = z^2 = 1, \quad xz = zx, \quad yz = zy. \quad (5)$$

Notice that adding to (5) the relation

$$xy = yx = z \quad (6)$$

we obtain the Klein group K_4 .

The Klein group $K_4 = \{1, x, y, z\}$ with the generators satisfying (5)-(6) is abelian. Thus, all its irreducible representations are one-dimensional. It implies that the minimal dimension of any faithful representation is equal to four. Pick up the orthonormal basis $\{|j\rangle, 1 \leq j \leq 4\}$ in the Hilbert space \mathcal{H}_4 , $\dim \mathcal{H}_4 = 4$. Then, we can define the standard faithful representation of K_4 in \mathcal{H}_4 by the formula

$$x|1\rangle = y|1\rangle = z|1\rangle = |1\rangle,$$

$$\begin{aligned}
x|2\rangle &= y|2\rangle = -|2\rangle, \quad z|2\rangle = |2\rangle, \\
x|3\rangle &= |3\rangle, \quad y|3\rangle = z|3\rangle = -|3\rangle, \\
x|4\rangle &= -|4\rangle, \quad y|4\rangle = |4\rangle, \quad z|4\rangle = -|4\rangle.
\end{aligned} \tag{7}$$

The quantum channel corresponding to the graph generated by the elements $\{1, x, y, z\}$ satisfying (7) is given by the formula

$$\Phi(\rho) = (1 - \alpha - \beta)\rho + \alpha x\rho x + \beta y\rho y, \tag{8}$$

$\rho \in \mathfrak{S}(H_4)$, $\alpha, \beta \geq 0$, $\alpha + \beta \leq 1$. The map defined by (8) is a dephasing channel which is a partial case of pseudo-diagonal channel (4) such that

$$\begin{aligned}
\Phi(|j\rangle\langle j|) &= |j\rangle\langle j|, \quad 1 \leq j \leq 4, \\
\Phi(|1\rangle\langle 2|) &= (1 - 2\alpha - 2\beta)|1\rangle\langle 2|, \quad \Phi(|1\rangle\langle 3|) = (1 - 2\beta)|1\rangle\langle 3|, \\
\Phi(|1\rangle\langle 4|) &= (1 - 2\alpha)|1\rangle\langle 4|, \quad \Phi(|2\rangle\langle 3|) = (1 - 2\alpha)|2\rangle\langle 3|, \\
\Phi(|2\rangle\langle 4|) &= (1 - 2\beta)|2\rangle\langle 4|, \quad \Phi(|3\rangle\langle 4|) = (1 - 2\alpha - 2\beta)|3\rangle\langle 4|.
\end{aligned}$$

A transition from the graph determined by (5)-(6) corresponding to the channel (8) to the graph \mathcal{L}_θ corresponding to a general pseudo-diagonal channel (4) can be considered as a non-commutative deformation of the Klein group K_4 in the spirit of [7].

Our goal is a explanation of subalgebras \mathcal{M}_θ of $Mat_4(\mathbb{C})$ generated by subspaces \mathcal{L}_θ , $\theta \in \mathbb{C}^*$ in terms of representation theory. It follows from (3) that the generators of \mathcal{M}_θ can be pick up as follows:

$$X = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & 0 & \theta & 0 \\ 0 & 0 & 0 & 1/\theta \\ 1/\theta & 0 & 0 & 0 \\ 0 & \theta & 0 & 0 \end{pmatrix}, Z = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \tag{9}$$

Alternatively it is possible to pick up the following matrices as generators of \mathcal{M}_θ for $\theta \in \mathbb{C}^*$:

$$X = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, XYX = \begin{pmatrix} 0 & 0 & 1/\theta & 0 \\ 0 & 0 & 0 & \theta \\ \theta & 0 & 0 & 0 \\ 0 & 1/\theta & 0 & 0 \end{pmatrix}, Z = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \tag{10}$$

It results in $\mathcal{M}_\theta = \mathcal{M}_{1/\theta}$. Also, we have the following symmetry $\mathcal{M}_\theta = \mathcal{M}_{-\theta}$. Note that $\dim_{\mathbb{C}} \mathcal{M}_\theta = 8$ for $\theta \neq \pm 1$ and $\dim_{\mathbb{C}} \mathcal{M}_\theta = 4$ for $\theta = \pm 1$.

For explanation of this effect we will construct family of algebras \mathcal{A}_θ , $\theta \in \mathbb{C}^*$ as a quotient of $\mathbb{C}G$ by relation:

$$(xy + yx)z = (\theta + \theta^{-1}) \cdot 1 \tag{11}$$

One can see that we have canonical isomorphism: $\mathcal{A}_\theta \cong \mathcal{A}_{1/\theta}$. For construction of this family we will use a semidirect product of \mathbb{Z}_2 and $\mathbb{Z} \oplus \mathbb{Z}_2$ and its representations. Matrices X, Y, Z defines

representation ϕ of algebras \mathcal{A}_θ and $\mathcal{M}_\theta = \phi(\mathcal{A}_\theta)$. Family \mathcal{A}_θ has the following symmetry $\mathcal{A}_\theta \cong \mathcal{A}_{-\theta}, \theta \in \mathbb{C}^*$. Any algebra \mathcal{A}_θ has dimension 8. Our main result tells us that algebra \mathcal{A}_θ is isomorphic to $Mat_2(\mathbb{C}) \oplus Mat_2(\mathbb{C})$ for $\theta \neq \pm 1$. For $\theta = \pm 1$ algebras \mathcal{A}_θ have 4-dimensional radical J , and, quotient \mathcal{A}_θ/J is a direct sum of four copies of \mathbb{C} 's. Algebra \mathcal{A}_θ is "universal" for representation ϕ , i.e.

- we have factorization: for any $\theta \in \mathbb{C}^*$ representation $\phi : \mathbb{C}G \rightarrow Mat_4(\mathbb{C})$ is a composition $\mathbb{C}G \rightarrow \mathcal{A}_\theta \rightarrow Mat_4(\mathbb{C})$.
- Also, for $\theta \neq \pm 1$ map $\phi : \mathcal{A}_\theta \rightarrow \mathcal{M}_\theta$ is an isomorphism.

Our article is organized as follows. In section 2 we discuss matrices X, Y, Z and Klein group. In this section we remind the notion of Klein group and some property of representations of it. In section 3, we give the definition of group G . This group G is natural object for studying of $\mathcal{M}_\theta, \theta \in \mathbb{C}^*$. Also, we give preliminary definitions and results in group theory. In particular, we prove that group G is a semidirect product of \mathbb{Z}_2 and $\mathbb{Z} \oplus \mathbb{Z}_2$. One can find the exploration of center of group algebra $\mathbb{C}G$ in section 4. This exploration is important for studying of representations of algebra $\mathbb{C}G$. We give a natural description of irreducible $\mathbb{C}G$ -modules in terms of maximal commutative subalgebra $\mathbb{C}P \subset \mathbb{C}G$, where P is a maximal abelian subgroup of G . One can describe the connection between \mathcal{M}_θ and $\mathbb{C}G$ in the following manner: matrices X, Y, Z defines representation of the group G . In section 6 we prove that $\mathbb{C}G$ - representation ϕ defined by matrices X, Y, Z is semisimple, i.e. direct sum of irreducible $\mathbb{C}G$ - modules. Also, we introduce the family of algebras \mathcal{A}_θ . This family is a quotient of $\mathbb{C}G$ by relations (11). Family \mathcal{A}_θ plays an important role for studying of representation ϕ . Actually, morphism $\phi : \mathbb{C}G \rightarrow Mat_4(\mathbb{C})$ factorizes into composition: $\mathbb{C}G \rightarrow \mathcal{A}_\theta \rightarrow Mat_4(\mathbb{C})$. Thus, $\phi(\mathcal{A}_\theta) = \mathcal{M}_\theta$. We prove that if $\theta \neq \pm 1$ algebra \mathcal{A}_θ is isomorphic to direct sum of two copies of matrix algebras $Mat_2(\mathbb{C})$. In the case $\theta = \pm 1$, algebra \mathcal{A}_θ has 4-dimensional radical. This means that relation (11) is universal, i.e. for general $\theta \in \mathbb{C}^*$ representation ϕ provides isomorphism. In the case $\theta = \pm 1$, representation ϕ is not an isomorphism, because algebras \mathcal{A}_θ has radical. Since ϕ is semisimple, ϕ annihilate radical and we get 4-dimensional algebra \mathcal{M}_θ . Article has two appendices. In the beginning of appendix A, we give some classical definitions in homological algebra. In this section we study homological properties of irreducible $\mathbb{C}G$ -modules. In particular, we calculate extension group for irreducible $\mathbb{C}G$ -modules. In Appendix B, we recall some definitions and notions of noncommutative algebraic geometry. We introduce two kinds of noncommutative varieties for arbitrary finite generated algebra A : representation space $\mathbf{Rep}_n A = \text{Hom}_{alg}(A, Mat_n(\mathbb{C}))$ and moduli variety $\mathcal{M}_n A$ which is the quotient of $\mathbf{Rep}_n A$ by natural action of $GL_n(\mathbb{C})$. In this section we study connection between varieties $\mathbf{Rep}_2 \mathbb{C}P$ and $\mathbf{Rep}_2 \mathbb{C}G$ as well as $\mathcal{M}_2 \mathbb{C}P$ and $\mathcal{M}_2 \mathbb{C}G$.

2 Matrices X, Y, Z and Klein's group.

In this section we will introduce matrices X, Y, Z depending on θ and consider partial cases $\theta = \pm 1$. In this case we will show that these matrices define representation of Klein's group.

Consider matrices X, Y and Z from the formula (14). Denote by I the identity matrix. It follows that $\mathcal{L}_\theta = aI + bX + cY + dZ, a, b, c, d \in \mathbb{C}$ for any $\theta \in \mathbb{C}^*$. Moreover, X, Y, Z satisfy (5). These relations motivate us to interpret subspace \mathcal{L}_θ as an image of admissible 4-dimensional representations of the group G .

One can see that if $\theta = 1$, then X, Y, Z have the following view:

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad (12)$$

respectively. In this case our matrices are commuting. And hence, we have the following relations:

$$XY = YX, XZ = ZX, YZ = ZY, X^2 = Y^2 = Z^2 = 1. \quad (13)$$

Also, one can check that

$$Z = XY. \quad (14)$$

If we consider group with generators x, y satisfying to relations (13) and (14), we get the well-known Klein's group K_4 of order 4 isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, where first \mathbb{Z}_2 is generated by x , second one is generated by y . Thus, in the case of $\theta = 1$ matrices X, Y define representation ϕ of K_4 by rule: $\phi : x \mapsto X, y \mapsto Y$. Recall that representation ϕ is a direct sum of irreducible ones. As for any abelian group, all irreducible representation of K_4 is 1-dimensional. These one-dimensional representations (or characters) are elements of group $\text{Hom}(K_4, \mathbb{C}^*)$, where \mathbb{C}^* is an algebraic torus by the definition. By Pontryagin's duality, group $\text{Hom}(K_4, \mathbb{C}^*) \cong K_4$. Thus, we can describe characters in the following manner:

$$\chi_1 : x \mapsto 1, y \mapsto 1$$

$$\chi_x : x \mapsto -1, y \mapsto 1$$

$$\chi_y : x \mapsto 1, y \mapsto -1$$

$$\chi_{xy} : x \mapsto -1, y \mapsto -1$$

One can check that representation ϕ is isomorphic to direct sum $\chi_1 \oplus \chi_x \oplus \chi_y \oplus \chi_{xy}$. It means that there is a basis in which matrix X and Y are diagonal and matrices X and Y have eigenvalues $\chi_1(x), \chi_x(x), \chi_y(x), \chi_{xy}(x)$ and $\chi_1(y), \chi_x(y), \chi_y(y), \chi_{xy}(y)$ respectively.

Also, consider the case of $\theta = -1$. One can check that if $\theta = -1$ matrices X, Y, Z are commuting and $Z = -XY$. Of course, matrices $X, -Y$ as images of generators x, y of K_4 define representation $\rho : x \mapsto X, y \mapsto -Y$ which is similar to the case $\theta = 1$.

In the next sections we will generalize description of $\mathcal{L}_{\pm 1}$ to the case of arbitrary θ by means of representation theory methods.

3 Some facts from group theory and description of the group G .

In this section we will study group G with generators x, y, z and relations (5).

Firstly, recall the notion of a free group $F(S)$ with a set of generators S . Suppose that $S = \{s_1, \dots, s_k\}$. Denote J the set $S \cup S^{-1}$. Define *word* as a product of elements of J . Word can be simplified by omitting consequent symbols s and s^{-1} . Word which cannot be simplified is called *reduced*. The set of reduced words equipped by the operation of concatenation of words is said to be a free group $F(S)$ of *rank* k . Finite generated group G is a quotient of free group with arbitrary set of generators. We will say that group H has *presentation* $H = \langle S | R \rangle$ where S is a set of generators, R is a set of relations. One can use the following description of R . Any relation we will rewrite in the view: $r = 1$, where $r \in F(S)$. Group H is a quotient $F(S)/N(R)$, where $N(R)$ is a *normal closure* in free group $F(S)$ of the set of relations R , i.e. minimal normal subgroup of $F(S)$ containing R . Of course, presentation of fixed group H is not unique.

Recall the construction of free product of groups. Assume that we have two groups H_1, H_2 which have the following presentations: $H_1 = \langle S_1 | R_1 \rangle$ and $H_2 = \langle S_2 | R_2 \rangle$ with different sets of generators S_1 and S_2 . Define free product $H_1 * H_2$ as group with presentation $\langle S_1 \cup S_2 | R_1 \cup R_2 \rangle$. One can show that notion of free product is well-defined. Analogously, one can define free product $H_1 * \dots * H_k$ of the groups H_1, \dots, H_k .

Assume that $S = \{x, y, z\}$. One can consider group G as quotient of $F(S)$. There are relations $x^2 = y^2 = z^2 = 1, xz = zx, yz = zy$. Rewrite these relations in the following manner: $x^2 = y^2 = z^2 = 1, xzx^{-1}z^{-1} = yzy^{-1}z^{-1} = 1$. Thus, $R = \{x^2, y^2, z^2, xzx^{-1}z^{-1}, yzy^{-1}z^{-1}\}$. Consider set of generators $S = \{x, y, z\}$, set of relations: $R_1 = \{x^2, y^2, z^2\} \subset R$, normal subgroup $N(R_1)$ and quotient $P = F(S)/N(R_1)$. The group P has the following presentation: $P = \langle S | R_1 \rangle$. It is clear that

$$P = \langle x, y, z | x^2, y^2, z^2 \rangle = \langle x | x^2 \rangle * \langle y | y^2 \rangle * \langle z | z^2 \rangle \cong \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2 \quad (15)$$

where \mathbb{Z}_2 is a cyclic group of order 2. In this way we get that group G is a quotient of P by the relations $xzx^{-1}z^{-1} = 1, yzy^{-1}z^{-1} = 1$.

Consider natural morphism: $\phi : \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2 \rightarrow \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Using results on subgroup of free product (independently Kurosh [8], Baer and Levi [9], Takahasi [10]), we get that kernel of ϕ is a free group F_5 of rank 5. It will be convenient to choose generators of the $\text{Ker} \phi = F_5$ as follows: $xyx^{-1}y^{-1}, xzx^{-1}z^{-1}, yzy^{-1}z^{-1}, xzyzy^{-1}z^{-1}x^{-1}, yxzx^{-1}z^{-1}y^{-1}$. Of course, using relations $x^2 = y^2 = z^2 = 1$, we obtain that $xyx^{-1}y^{-1} = xyxy, xzx^{-1}z^{-1} = xzxz, yzy^{-1}z^{-1} = yzyz, xzyzy^{-1}z^{-1}x^{-1} = xyzzyx, yxzx^{-1}z^{-1}y^{-1} = yxzxzy$. One can show that group G is a quotient of P by the relations $R_2 = \{xzx^{-1}z^{-1} = xzxz = 1, yzy^{-1}z^{-1} = yzyz = 1\}$. Consider normal closure $N(R_2)$ in the group P of the set of the elements R_2 . Consider quotient of $\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$ by normal subgroup $N(R_2)$.

Let us prove the following fact:

Proposition 1. *We have the following exact sequence for non-abelian group G :*

$$0 \longrightarrow \mathbb{Z} \longrightarrow G \longrightarrow \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \longrightarrow 1, \quad (16)$$

where normal subgroup \mathbb{Z} of the group G is generated by element $xyxy$.

Proof. We have the following exact sequence for free product $\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$:

$$1 \longrightarrow F_5 \longrightarrow \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2 \longrightarrow \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \longrightarrow 1, \quad (17)$$

As we know, free group F_5 is generated by $xyxy, xzxz, yzyz, xzyzzy, yxzxzy$. Group $N(R_2)$ is a normal closure of $xzxz, yzyz$. If $xzxz = 1$ and $yzyz = 1$ then $xzyzzy = 1$ and $yxzxzy = 1$. Denote by a_1, \dots, a_5 the generators $xyxy, xzxz, yzyz, xzyzzy, yxzxzy$ of F_5 . Denote by $F'_5 = [F_5, F_5]$ commutant of F_5 . Consider natural morphism $f : F_5 \rightarrow F_5/F'_5 = \mathbb{Z}^{\oplus 5}$. Denote by $a_i, i = 1, \dots, 5$ the generators of $\mathbb{Z}^{\oplus 5}$ which are images of $xyxy, xzxz, yzyz, xzyzzy, yxzxzy$ under f respectively. Also, we have projection $\mathbb{Z}^{\oplus 5} \rightarrow \mathbb{Z}$ defined by rule: $\sum k_i a_i \mapsto k_1 a_1$. Consider composition $F_5 \rightarrow \mathbb{Z}$. One can check that $N(R_2)$ is a kernel of this composition.

Moreover, we get the following commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & G & \longrightarrow & \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \longrightarrow 1 \\ & & \uparrow & & \uparrow & & \cong \uparrow \\ 1 & \longrightarrow & F_5 & \longrightarrow & \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2 & \longrightarrow & \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \longrightarrow 1 \\ & & \uparrow & & \uparrow & & \uparrow \\ 1 & \longrightarrow & N(R_2) & \longrightarrow & N(R_2) & \longrightarrow & 1 \end{array} \quad (18)$$

□

Exact sequence (16) tells us that group G is the extension of $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ by \mathbb{Z} . Automorphism group $\text{Aut}(\mathbb{Z}) \cong \mathbb{Z}_2$, i.e. there are only two automorphisms of \mathbb{Z} : trivial and morphism defined by correspondence: $n \mapsto -n, n \in \mathbb{Z}$. It is clear that we have a well-defined action of $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ on \mathbb{Z} by conjugation. This action defines homomorphism of groups: $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \rightarrow \text{Aut}(\mathbb{Z}) = \mathbb{Z}_2$. One can calculate that kernel of this morphism is a subgroup $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ generated by images of xy and z under natural morphism $G \rightarrow \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Denote by P subgroup of G generated by xy and z . We have the following commutative diagram:

$$\begin{array}{ccccccc} & & \mathbb{Z}_2 & \xrightarrow{=} & \mathbb{Z}_2 & \longrightarrow & 1 \\ & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & G & \longrightarrow & \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \longrightarrow 1 \\ & & \uparrow & & \uparrow & & \uparrow \\ & & \mathbb{Z} & \xrightarrow{=} & P & \longrightarrow & \mathbb{Z}_2 \oplus \mathbb{Z}_2 \longrightarrow 1 \end{array} \quad (19)$$

It follows that group P is abelian. Also, one can show that $P \cong \mathbb{Z} \oplus \mathbb{Z}_2$.

Recall the notion of *semidirect* product. Consider group H . Assume that there are normal subgroup N_1 and subgroup H_1 of H . Also, assume that composition of embedding $H_1 \hookrightarrow H$ and projection $H \rightarrow H/N_1$ is an isomorphism $H_1 \cong H/N_1$. In this case, we will say that H is

a semidirect product of N_1 and H_1 . We will denote by $N_1 \rtimes H_1$ the semidirect product of N_1 and H_1 . Equivalently, assume that there is an exact sequence:

$$1 \longrightarrow N_1 \longrightarrow H \longrightarrow H/N_1 \longrightarrow 1. \quad (20)$$

H is a semidirect product of N_1 and H_1 iff sequence (20) is split, i.e. there is a section $H/N_1 \rightarrow H$ of the natural morphism $H \rightarrow H/N_1$. Also, one can say that there is a subgroup H_1 such that $H_1 \cong H/N_1$ and $H_1 \cap N_1 = \{1\}$.

Proposition 2. *Group G is a semidirect product $(\mathbb{Z} \oplus \mathbb{Z}_2) \rtimes \mathbb{Z}_2$.*

Proof. We have the following exact sequence:

$$0 \longrightarrow P = \mathbb{Z} \oplus \mathbb{Z}_2 \longrightarrow G \longrightarrow \mathbb{Z}_2 \longrightarrow 1 \quad (21)$$

Recall that subgroup $P = \mathbb{Z} \oplus \mathbb{Z}_2$ is generated by xy and z . This sequence has section $\mathbb{Z}_2 \rightarrow G$ defined by element x . Therefore, G is a semidirect product $(\mathbb{Z} \oplus \mathbb{Z}_2) \rtimes \mathbb{Z}_2$. \square

4 Center of group algebra $\mathbb{C}G$.

Let us pick up generators of group $G = (\mathbb{Z} \oplus \mathbb{Z}_2) \rtimes \mathbb{Z}_2$ as follows: $g = xy$, x and z . Elements g, x, z are generators because we have the identity: $y = xg$. Element g has infinite order. In this case group $P = \mathbb{Z} \oplus \mathbb{Z}_2$ is generated by g and z . We will use multiplicative description of \mathbb{Z} . Thus, we have the following relations: $x^2 = z^2 = 1, xz = zx, zg = gz, xgx = g^{-1}$. In this section we will study representation theory of group G .

Since group P is a subgroup of index 2 in the group G , we can formulate the following useful fact:

Proposition 3. *Any element $w \in G$ can be uniquely written as follows: $w = g'$ or $w = g'x$ for arbitrary element $g' \in P$.*

Recall the notion of free A -module for arbitrary algebra A . A -module M is a free A -module of rank n iff M is isomorphic to direct sum of n copies of A : $M \cong \bigoplus_{i=1}^n A$.

Corollary 4. *Group algebra $\mathbb{C}G$ is a free $\mathbb{C}P$ - module of rank 2.*

Proof. Using decomposition $G = P \cup Px$, we get the decomposition of $\mathbb{C}P$ -module $\mathbb{C}G$ into direct sum $\mathbb{C}P \oplus \mathbb{C}P$. Also, we can choose element $1, x \in G$ as generators of these free $\mathbb{C}P$ -modules. \square

Group algebra $\mathbb{C}P$ is an extended algebra of Laurent's polynomials $\mathbb{C}[g, g^{-1}, z; z^2 = 1]$. Consider center \mathcal{C} of group algebra $\mathbb{C}G$. Center \mathcal{C} is a subalgebra of $\mathbb{C}P$ containing elements commuting with x . Equivalently, element $c \in \mathcal{C}$ iff $xcx^{-1} = xc = c$. As we know, we have well-defined action of G on P by conjugation. Therefore, we have involution c_x defined by rule:

$$c_x : (g, z) \mapsto (xgx^{-1}, xzx^{-1}) = (xgx, xzx) = (g^{-1}, z). \quad (22)$$

Using this involution, we get that \mathcal{C} is a subalgebra of $\mathbb{C}G$ consisting of c_x - invariant elements.

It can be shown that \mathcal{C} is a subalgebra of $\mathbb{C}P$ generated by $g + g^{-1}$ and z . Thus,

$$\mathcal{C} \cong \mathbb{C}[u = g + g^{-1}] \otimes \mathbb{C}[z, z^2 = 1]. \quad (23)$$

Consider $\mathbb{C}P$ as \mathcal{C} - module. Prove the following useful fact.

Proposition 5. $\mathbb{C}P$ is a free \mathcal{C} - module of rank 2.

Proof. It is sufficient to prove that $\mathbb{C}[g, g^{-1}]$ is a free \mathcal{C} -module of rank 2. For this purpose, consider fixed polynomial $f(g) \in \mathbb{C}[g, g^{-1}]$. Prove that there are uniquely determined elements $z_1, z_2 \in \mathcal{C}$ such that

$$f(g) = z_1 + gz_2. \quad (24)$$

Multiply this identity by $x \in G$ from both sides. We get the following identity:

$$xf(g)x = f(g^{-1}) = z_1 + g^{-1}z_2 \quad (25)$$

Thus, we get that

$$f(g) - f(g^{-1}) = (g - g^{-1})z_2. \quad (26)$$

It follows that $z_2 = \frac{f(g) - f(g^{-1})}{g - g^{-1}} \in \mathcal{C}$ is a Laurent's polynomial over g . Also, $z_1 = f(g) - gz_2$. Therefore, elements z_1 and z_2 are uniquely determined by $f(g)$ and, hence, $\mathbb{C}P$ is a free \mathcal{C} - module of rank 2. \square

Corollary 6. Algebra $\mathbb{C}G$ is a free \mathcal{C} - module of rank 4.

Remind the following useful notion of tensor product of modules over algebra. Let V_1 and V_2 be right and left A -module respectively for some algebra A . Denote by $V_1 \otimes_A V_2$ the quotient of $V_1 \otimes V_2$ by relations $v_1a \otimes v_2 - v_1 \otimes av_2$ for any $v_1 \in V_1, v_2 \in V_2, a \in A$. If V_1 has structure of left A - module, then $V_1 \otimes_A V_2$ is A -module.

For fixed algebra and its representation, the space of representation is a module. If representation is irreducible, then we will say that corresponding module is irreducible. Further, let us study irreducible $\mathbb{C}G$ - modules.

Since \mathcal{C} is a commutative algebra, we can consider $\mathbb{C}G$ as algebra over \mathcal{C} . Fix character $\chi \in \text{Spec}\mathcal{C}$. Denote \mathcal{C} -module corresponding to χ by \mathbb{C}^χ . It can be shown that $\mathbb{C}G \otimes_{\mathcal{C}} \mathbb{C}^\chi$ is an algebra over \mathbb{C} . One can check that algebra $\mathbb{C}G \otimes_{\mathcal{C}} \mathbb{C}^\chi$ is a quotient of $\mathbb{C}G$ by relations $z - \chi(z) \cdot 1$.

By Schur's lemma, for any central element $c \in \mathcal{C}$ and any irreducible representation ρ of algebra $\mathbb{C}G$ matrix $\rho(c)$ is scalar, i.e. $\rho(c) = \chi(c)1$ for some character χ of \mathcal{C} . Thus, we get that any irreducible $\mathbb{C}G$ - module correspond to some character of \mathcal{C} . Set of characters of \mathcal{C} is a variety $\text{Spec}\mathcal{C} = \text{Hom}_{\text{alg}}(\mathcal{C}, \mathbb{C})$. This variety is called *variety of characters* of the algebra \mathcal{C} .

Let us give some description of characters of \mathcal{C} . Any $f \in \text{Hom}_{\text{alg}}(\mathcal{C}, \mathbb{C})$ is defined by pair of values $f(u) = f(g + g^{-1})$ and $f(z)$. One can show that $f(u) \in \mathbb{C}$ and because of $z^2 = 1$ we get that $f(z) = \pm 1$. Thus, $\text{Spec}\mathcal{C}$ has two components. Each of them is an affine line \mathbb{C}^1 . Denote these components by \mathbb{C}_-^1 and \mathbb{C}_+^1 corresponding to values $f(z) = -1$ and $f(z) = 1$ respectively.

All irreducible $\mathbb{C}G$ -modules correspond to fixed character χ are representations of $\mathbb{C}G \otimes_{\mathbb{C}} \mathbb{C}^{\chi}$. Thus, we have to study algebra $\mathbb{C}G \otimes_{\mathbb{C}} \mathbb{C}^{\chi}$ for description of irreducible $\mathbb{C}G$ - modules. For studying of algebra $\mathbb{C}G \otimes_{\mathbb{C}} \mathbb{C}^{\chi}$, let us remind that Jacobson radical (or radical) of the algebra A is an intersection of maximal right ideals of A . Equivalent formulation: Jacobson radical consists of all elements annihilated by all simple left A -modules. Also, note that for finite-dimensional algebras we have the following description: Jacobson radical is a maximal nilpotent ideal.

Proposition 7. *If $\chi(u) = \chi(g + g^{-1}) \neq \pm 2$, then algebra $\mathbb{C}G \otimes_{\mathbb{C}} \mathbb{C}^{\chi}$ is isomorphic to $\text{Mat}_2(\mathbb{C})$. If $\chi(u) = \chi(g + g^{-1}) = \pm 2$, then algebra $\mathbb{C}G \otimes_{\mathbb{C}} \mathbb{C}^{\chi}$ has 2-dimensional radical.*

Proof. Using corollary 6, we get that $\dim_{\mathbb{C}} \mathbb{C}G \otimes_{\mathbb{C}} \mathbb{C}^{\chi} = 4$. It can be shown that this algebra has the following basis: $1, x, g, xg$. There are relations: $x^2 = 1, g + g^{-1} = \chi(u)1, xgx = g^{-1}, \chi(u) \in \mathbb{C}$. Denote by t the root of equation: $t + t^{-1} = \chi(u)$. We have the following morphism: $F : \mathbb{C}G \otimes_{\mathbb{C}} \mathbb{C}^{\chi} \rightarrow \text{Mat}_2(\mathbb{C})$ defined by rule:

$$F : x \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, g \mapsto \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \quad (27)$$

One can check that if $\chi(u) \neq \pm 2$, then F is an isomorphism. If $\chi(u) = 2$, then $g + g^{-1} = 2$ and, hence, $(g - 1)^2 = 0$. Consider two-sided ideal J of $\mathbb{C}G \otimes_{\mathbb{C}} \mathbb{C}^{\chi}$ generated by $g - 1$. It can be shown that J has a basis $x(g - 1), g - 1$ over \mathbb{C} . Also, consider $J^2 = \langle j_1 \cdot j_2 | j_1, j_2 \in J \rangle$. One can check that $J^2 = 0$. Thus, $\dim_{\mathbb{C}} J = 2$. Note that element x and 1 are not in Jacobson radical, because these element are not nilpotent. Thus, J is a maximal nilpotent ideal of $\mathbb{C}G \otimes_{\mathbb{C}} \mathbb{C}^{\chi}$. Therefore, J is a Jacobson radical. \square

Corollary 8. *Dimension of irreducible representation of group G is less or equal to 2.*

Proof. Irreducible representations of $\mathbb{C}G$ are irreducible representations of $\mathbb{C}G \otimes_{\mathbb{C}} \mathbb{C}^{\chi}$ for some $\chi \in \text{Spec} \mathcal{C}$. Using proposition 7, we get that irreducible representations of $\mathbb{C}G \otimes_{\mathbb{C}} \mathbb{C}^{\chi}$ is two-dimensional if $\chi(u) \neq \pm 2$. If $\chi(u) = \pm 2$, then irreducible representations of $\mathbb{C}G$ are one-dimensional. \square

5 Description and properties of irreducible $\mathbb{C}G$ - modules.

In this section we will describe irreducible $\mathbb{C}G$ - modules in terms of $\mathbb{C}P$ - modules.

Assume that V is an irreducible $\mathbb{C}G$ - module. It follows from Corollary 8 that $\dim_{\mathbb{C}} V \leq 2$. Also, recall that algebra $\mathbb{C}P$ is an extended algebra of Laurent polynomials: $\mathbb{C}[g^{\pm 1}] \otimes \mathbb{C}[z, z^2 = 1]$.

Consider variety $\text{Spec} \mathbb{C}P = \text{Hom}_{\text{alg}}(\mathbb{C}[g^{\pm 1}] \otimes \mathbb{C}[z, z^2 = 1], \mathbb{C})$. Then, $\chi(z) = \pm 1$ for any $\chi \in \text{Spec} \mathbb{C}P$. Thus, variety $\text{Spec} \mathbb{C}P$ has two components, each of them is an one-dimensional algebraic torus \mathbb{C}^* . Components correspond to $\chi(z) = \pm 1$. We will denote these components by \mathbb{C}_-^* and \mathbb{C}_+^* . We have the following description of $\chi \in \text{Spec} \mathbb{C}P$: $\chi(g, z) = (\chi(g) = a, \chi(z) = \pm 1) \in \mathbb{C}^* \times \{\pm 1\}$. Remind that \mathcal{C} is an algebra of invariants of $\mathbb{C}P$ under action of involution c_x

defined by rule (22). Also, with action of c_x on \mathbb{CP} , we have induced action of c_x on $\text{Spec}\mathbb{CP}$. For character $\chi \in \text{Spec}\mathbb{CP}$, we have the following identity: $c_x(\chi)(g) = \chi(xgx) = \chi(g^{-1}) = \chi^{-1}(g)$, i.e. $c_x(\chi) = \chi^{-1}$. We have the following surjective morphism of varieties:

$$pr : \text{Spec}\mathbb{CP} \rightarrow \text{Spec}\mathcal{C} \quad (28)$$

defined by rule: $pr : (a, 1) \mapsto (a + a^{-1}, 1)$ and $pr : (a, -1) \mapsto (a + a^{-1}, -1)$. One can check that involution c_x acts on the fibers of the morphism pr . Also, one can say that variety $\text{Spec}\mathcal{C}$ is a quotient of $\text{Spec}\mathbb{CP}$ by involution c_x . Recall that $\text{Spec}\mathbb{CP}$ and $\text{Spec}\mathcal{C}$ are disjoint unions $\mathbb{C}_-^* \cup \mathbb{C}_+^*$ and $\mathbb{C}_-^1 \cup \mathbb{C}_+^1$ respectively. Components \mathbb{C}_\pm^* and \mathbb{C}_\pm^1 correspond to different values of $\chi(z) = \pm 1$. Morphism pr decomposes into $pr : \mathbb{C}_-^* \rightarrow \mathbb{C}_-^1$ and $pr : \mathbb{C}_+^* \rightarrow \mathbb{C}_+^1$.

Consider ideal I of \mathbb{CP} generated by elements $(g - \chi(g) \cdot 1)^k$ and $z - \chi(z) \cdot 1$. Denote by $\mathbb{C}^\chi(k)$ the quotient of \mathbb{CP} -module \mathbb{CP}/I . It follows that $\mathbb{C}^\chi = \mathbb{C}^\chi(1)$. Consider finite-dimensional \mathbb{CG} - module V . Denote by $V_\chi(k)$ the following subspace of V :

$$V_\chi(k) = \{v \in V | (g - \chi(g) \cdot 1)^k v = 0, (z - \chi(z) \cdot 1)v = 0\} \quad (29)$$

It follows that $V_\chi(k) \subseteq V_\chi(k+1)$ for any integer k . Also, note that if $V_\chi(1) = 0$ then $V_\chi(k) = 0$ for any k . Actually, if we consider restriction of g on $V_\chi(k)$, then this restriction has eigenvector. Thus, if $V_\chi(k) \neq 0$ then $V_\chi(1) \neq 0$.

Since V is finite-dimensional, we get that there is a minimal k_0 such that $V_\chi(k_0) = V_\chi(m)$ for $m \geq k_0$. Denote by $\text{Char}(V)$ the set of characters $\chi \in \text{Spec}\mathbb{CP}$ such that $V_\chi(1) \neq 0$.

We have the following famous fact:

Proposition 9. • Consider \mathbb{CP} -module V such that $\text{Char}(V) = \{\chi\}$. Then there is a decomposition

$$V = \mathbb{C}^\chi(k_1) \oplus \dots \oplus \mathbb{C}^\chi(k_q). \quad (30)$$

for arbitrary integers k_1, \dots, k_q .

• Consider finite-dimensional \mathbb{CP} - module W . There are \mathbb{CP} -modules V_1, \dots, V_s and characters χ_1, \dots, χ_s such that

$$W \cong V_1 \oplus \dots \oplus V_s \quad (31)$$

and $\text{Char}V_i = \{\chi_i\}, i = 1, \dots, s$.

Proposition 10. Consider finite-dimensional \mathbb{CG} - module V . We have the following identity:

$$\dim_{\mathbb{C}} V_\chi(k) = \dim_{\mathbb{C}} V_{c_x(\chi)}(k)$$

for any k . In particular, $\chi \in \text{Char}(V)$ iff $x(\chi) = \chi^{-1} \in \text{Char}(V)$ and $\text{Char}(\mathbb{CG} \otimes_{\mathbb{CP}} \mathbb{C}^\chi) = (\chi, \chi^{-1})$

Proof. Assume that $v \in V_\chi(k)$. Consider vector xv . We have the following identities:

$$x(g - \chi(g) \cdot 1)^k v = (g^{-1} - \chi(g) \cdot 1)^k xv = (-1)^k g^{-k} (g - \chi^{-1}(g) \cdot 1)^k xv = 0 \quad (32)$$

Thus, $x(V_\chi(k)) = V_{c_x(\chi)}(k)$. One can check that $\text{Char}(\mathbb{CG} \otimes_{\mathbb{CP}} \mathbb{C}^\chi) = (\chi, \chi^{-1})$. In fact, assume that $v \in \mathbb{C}^\chi$ such that $gv = \chi(g)v$. Thus, we can choose the following basis of $\mathbb{CG} \otimes_{\mathbb{CP}} \mathbb{C}^\chi$: $1 \otimes v$ and $x \otimes v$. And one can check that V_χ and $V_{c_x(\chi)}$ are one-dimensional spaces $\mathbb{C}(1 \otimes v)$ and $\mathbb{C}(x \otimes v)$ respectively. \square

Corollary 11. *We have the following isomorphism: $\mathbb{C}G \otimes_{\mathbb{C}P} \mathbb{C}^{x(\chi)} \cong \mathbb{C}G \otimes_{\mathbb{C}P} \mathbb{C}^\chi$. Also, $\mathbb{C}G \otimes_{\mathbb{C}P} \mathbb{C}^\chi$ as $\mathbb{C}P$ -module is isomorphic to direct sum $\mathbb{C}^\chi \oplus \mathbb{C}^{c_x(\chi)}$.*

Thus, we can deduce the following

Corollary 12. *Consider irreducible $\mathbb{C}G$ - module V . Assume that $\chi \in \text{Char}(V)$ and $\chi(g) \neq \pm 1$. Then $\mathbb{C}G$ - module V is isomorphic to $\mathbb{C}G \otimes_{\mathbb{C}P} \mathbb{C}^\chi$.*

Proof. Firstly, consider $V = \mathbb{C}G \otimes_{\mathbb{C}P} \mathbb{C}^\chi$. Assume that $W \subseteq V$ is a $\mathbb{C}G$ - submodule. Consider $\text{Char}(W)$. Clearly, $\text{Char}(W) \subseteq \text{Char}(V)$. Since $\chi(g) \neq \pm 1$, then $c_x(\chi)(g) \neq \chi(g)$, i.e. $c_x(\chi) = \chi^{-1} \neq \chi$. Using proposition 10, we get that $\text{Char}(W) = \text{Char}(V)$ and, hence, $\dim_{\mathbb{C}} W = 2$. Thus, $W = V$.

Conversely, let V be irreducible $\mathbb{C}G$ - module. As we know from corollary 8, $\dim_{\mathbb{C}} V \leq 2$. If $\chi(g) \neq \pm 1$, then $\chi^{-1} \neq \chi$ and $\text{Char}(V) = (\chi, \chi^{-1})$. Thus, $\dim_{\mathbb{C}} V = 2$. Let v be a vector such that $gv = \chi(g)v$. One can show that v and xv generate $\mathbb{C}G$ - submodule isomorphic to $\mathbb{C}G \otimes_{\mathbb{C}P} \mathbb{C}^\chi$. Using irreducibility of V , we get the required. \square

Consider the case $\chi(g) = \pm 1$. We can prove the following corollary:

Corollary 13. *Assume that $\chi(g) = \pm 1$. In this case $\mathbb{C}G$ - module $V = \mathbb{C}G \otimes_{\mathbb{C}P} \mathbb{C}^\chi$ is a direct sum of one dimensional modules: $\mathbb{C}_1^\chi \oplus \mathbb{C}_{-1}^\chi$, where \mathbb{C}_1^χ and \mathbb{C}_{-1}^χ are subspaces generated by eigenvector of x corresponding to eigenvalues 1 and -1 respectively.*

Proof. In the case $\chi(g) = \pm 1$, group P acts on V by scalar matrices, and hence, any subspace is invariant under action of P . Space V has the following basis $1 \otimes v$, $x \otimes v$, where v is a basis of one-dimensional $\mathbb{C}P$ - module \mathbb{C}^χ . One can show that \mathbb{C}_1^χ and \mathbb{C}_{-1}^χ are one-dimensional subspaces generated by $1 \otimes v + x \otimes v$ and $1 \otimes v - x \otimes v$ respectively. Actually, $x(1 \otimes v + x \otimes v) = x(1+x) \otimes v = (x+x^2) \otimes v = (1+x) \otimes v$ and $x(1 \otimes v - x \otimes v) = x(1-x) \otimes v = -(1-x) \otimes v$. \square

6 Space \mathcal{L}_θ , representations of P and deformations of Klein group.

Let us come back to subspace $\mathcal{L}_\theta \subset M_4(\mathbb{C})$ generated by matrices X, Y, Z .

Proposition 14. • *For fixed $\theta \in \mathbb{C}^*$, $\theta \neq \pm 1$ representation ϕ of group $G = (\mathbb{Z} \oplus \mathbb{Z}_2) \rtimes \mathbb{Z}_2 = P \rtimes \mathbb{Z}_2$ defined by matrices X, Y, Z is isomorphic to $\mathbb{C}G \otimes_{\mathbb{C}P} \mathbb{C}^{\chi_1} \oplus \mathbb{C}G \otimes_{\mathbb{C}P} \mathbb{C}^{\chi_2}$, where $\chi_1 = (\chi_1(g) = \theta, \chi_1(z) = 1)$ and $\chi_2 = (\chi_2(g) = -\theta, \chi_2(z) = -1)$.*

• *In the case $\theta = \pm 1$ matrices X, Y, Z define representation of G : $\mathbb{C}_1^{\chi_1} \oplus \mathbb{C}_{-1}^{\chi_1} \oplus \mathbb{C}_1^{\chi_2} \oplus \mathbb{C}_{-1}^{\chi_2}$, where $\chi_1 = (\chi_1(g) = \theta, \chi_1(z) = 1)$ and $\chi_2 = (\chi_2(g) = -\theta, \chi_2(z) = -1)$.*

Proof. Using described early technics, representation of $\mathbb{C}G$ is given by representation of $\mathbb{C}P$. Consider matrices XY and Z .

$$XY = \begin{pmatrix} 0 & 0 & 0 & 1/\theta \\ 0 & 0 & \theta & 0 \\ 0 & \theta & 0 & 0 \\ 1/\theta & 0 & 0 & 0 \end{pmatrix}, Z = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad (33)$$

One can show that there is a basis in which matrix XY and Z have the following view:

$$XY = \begin{pmatrix} 1/\theta & 0 & 0 & 0 \\ 0 & \theta & 0 & 0 \\ 0 & 0 & -1/\theta & 0 \\ 0 & 0 & 0 & -\theta \end{pmatrix}, Z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad (34)$$

Let V be a 4-dimensional $\mathbb{C}G$ - module given by matrices X, Y, Z . If $\theta \neq \pm 1$, then we have two irreducible submodules $\mathbb{C}G \otimes_{\mathbb{C}P} \mathbb{C}^{\chi_1}$ and $\mathbb{C}G \otimes_{\mathbb{C}P} \mathbb{C}^{\chi_2}$ of M , where $\chi_1 = (\chi_1(g) = \theta, \chi_1(z) = 1)$ and $\chi_2 = (\chi_2(g) = -\theta, \chi_2(z) = -1)$.

Therefore, we have the following commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{C}G \otimes_{\mathbb{C}P} \mathbb{C}^{\chi_1} & \longrightarrow & V & \longrightarrow & W \longrightarrow 0 \\ & & & & \uparrow & & \\ & & & & \mathbb{C}G \otimes_{\mathbb{C}P} \mathbb{C}^{\chi_2} & & \end{array} \quad (35)$$

Since $\mathbb{C}G \otimes_{\mathbb{C}P} \mathbb{C}^{\chi_1}$ and $\mathbb{C}G \otimes_{\mathbb{C}P} \mathbb{C}^{\chi_2}$ are not isomorphic and irreducible, then composition $\mathbb{C}G \otimes_{\mathbb{C}P} \mathbb{C}^{\chi_2} \rightarrow V \rightarrow W$ is an immersion. Also, note that $\dim_{\mathbb{C}} W = \dim_{\mathbb{C}} \mathbb{C}G \otimes_{\mathbb{C}P} \mathbb{C}^{\chi_2} = 2$. Thus, $\mathbb{C}G \otimes_{\mathbb{C}P} \mathbb{C}^{\chi_2} \cong W$. Thus, $\mathbb{C}G$ -module V is a direct sum $\mathbb{C}G \otimes_{\mathbb{C}P} \mathbb{C}^{\chi_1} \oplus \mathbb{C}G \otimes_{\mathbb{C}P} \mathbb{C}^{\chi_2}$.

As we know from section 2 if $\theta = \pm 1$, then representation ϕ defines regular representation of Klein group. \square

For further studying of representations of G from proposition, consider $\mathbb{C}G$ - module V corresponding to representation of G given by matrices X, Y, Z . Using theorem, we get that $\text{Char}(V) = \{(\chi(g) = \theta, \chi(z) = 1), (\chi(g) = 1/\theta, \chi(z) = 1), (\chi(g) = -\theta, \chi(z) = -1), (\chi(g) = -1/\theta, \chi(z) = -1)\}$. One can define the automorphism s of group algebra (not group!) $\mathbb{C}P$ defined by formula:

$$g \mapsto -g, z \mapsto -z. \quad (36)$$

Also, we have the involution $c_x : (g, z) \mapsto (g^{-1}, z)$ defined early. One can check that automorphisms s and c_x commute. Thus, we have an action of $\mathbb{Z}_2 \times \mathbb{Z}_2$ on group algebra $\mathbb{C}P$. Consider algebra of invariants $\mathbb{C}P^{\mathbb{Z}_2 \times \mathbb{Z}_2} = \{a \in \mathbb{C}P \mid c_x(a) = a, s(a) = a\}$.

Proposition 15. *Algebra of invariants $\mathbb{C}P^{\mathbb{Z}_2 \times \mathbb{Z}_2}$ is generated by elements $v = (g + g^{-1})z$. Also, $\mathbb{C}P$ is a free $\mathbb{C}P^{\mathbb{Z}_2 \times \mathbb{Z}_2}$ - module of rank 4.*

Proof. It can be shown that $\mathbb{C}P^{\mathbb{Z}_2 \times \mathbb{Z}_2} = \mathcal{C}^s = \{p \in \mathcal{C} \mid s(p) = p\}$. Recall that \mathcal{C} is a quotient of $\mathbb{C}[u, z]$ by relation $z^2 = 1$. It is easy that relation $z^2 = 1$ is s -invariant. Thus, \mathcal{C}^s is a quotient of $\mathbb{C}[u, z]^s = \{p(u, z) = p(-u, -z)\}$ by relation $z^2 = 1$. It can be calculated that $\mathbb{C}[u, z]^s = \mathbb{C}[v_1 = u^2, v = uz, v_3 = z^2, v_1 v_3 = v^2]$. Using relation $z^2 = 1$, we get that $\mathcal{C}^s = \mathbb{C}[v_1, v, v_1 = v^2]$. Analogous to proposition 5, one can prove that $\mathbb{C}P$ is a free $\mathbb{C}P^{\mathbb{Z}_2 \times \mathbb{Z}_2}$ - module of rank 4. \square

Corollary 16. *$\mathbb{C}G$ is a free $\mathbb{C}[v]$ - module of rank 8.*

Fix character χ of \mathcal{C}^s . Denote by b the value $\chi(v)$. Denote by \mathbb{C}^χ the $\mathbb{C}[v]$ - module corresponding to χ . Also, we can consider the following algebra $\mathcal{A}_\theta = \mathbb{C}G \otimes_{\mathcal{C}^s} \mathbb{C}^\chi$. For fixed $b \in \mathbb{C}$ algebra \mathcal{A}_θ is a quotient of $\mathbb{C}G$ by relation:

$$(g + g^{-1})z = b \cdot 1, \quad (37)$$

here $b = \chi(z(g + g^{-1})) = \theta + \theta^{-1}$. Consider representation ϕ of G . We have the following commutative diagram:

$$\begin{array}{ccc} \mathbb{C}G & \xrightarrow{\phi} & \text{Mat}_4(\mathbb{C}) \\ & \searrow & \nearrow \\ & \mathcal{A}_\theta & \end{array} \quad (38)$$

It means that for any $\theta \in \mathbb{C}^*$ matrix $\phi(g + g^{-1})z = (XY + YX)Z$ is $(\theta + \theta^{-1})I$, where I is an identity matrix.

Let us prove the following

Theorem 17. *For fixed $\theta \in \mathbb{C}^*$ $\dim_{\mathbb{C}} \mathcal{A}_\theta = 8$.*

- If $b = \theta + \theta^{-1} \neq 0$ then algebra \mathcal{A}_θ has the following basis $1, g, g^2, g^3, x, xg, xg^2, xg^3$ and the relations:

$$x^2 = 1, g^4 + (2 - b^2)g^2 + 1 = 0, xgx = g^{-1} = (b^2 - 2)g - g^3, b = \theta + \theta^{-1}. \quad (39)$$

Also, we have canonical isomorphism: $\mathcal{A}_\theta \cong \mathcal{A}_{-\theta}$ for any $\theta \in \mathbb{C}, \theta \neq \pm i$.

- If $b = \theta + \theta^{-1} = 0$, then $\mathcal{A}_{\pm i}$ has the following basis $1, g, x, z, xg, xz, gz, xgz$ and the relations:

$$x^2 = z^2 = 1, g^2 = -1, xgx = g^{-1}, xz = zx, gz = zg \quad (40)$$

Also, we have the following description of algebra \mathcal{A}_θ for various $\theta \in \mathbb{C}^*$:

- If $\theta \neq \pm 1$, then $\mathcal{A}_\theta \cong \text{Mat}_2(\mathbb{C}) \oplus \text{Mat}_2(\mathbb{C})$.
- If $\theta = \pm 1$, then algebra $\mathcal{A}_{\pm 1}$ has 4-dimensional Jacobson radical J and $\mathcal{A}_{\pm 1}/J = \bigoplus_{i=1}^4 \mathbb{C}$.

Proof. Using corollary 16, we get that for fixed $\theta \in \mathbb{C}$ algebra \mathcal{A}_θ has dimension 8. Using relation $(g + g^{-1})z = b \cdot 1$, we get that $g + g^{-1} = b \cdot z$. If $b \neq 0$, then $z = \frac{1}{b}(g + g^{-1})$. Also, $b^2 \cdot z^2 = b^2 \cdot 1 = (g + g^{-1})^2$. Thus, we can express z in terms of g and since $\dim_{\mathbb{C}} \mathcal{A}_\theta = 8$, we obtain that \mathcal{A}_θ has the required basis and the relations. If $b = 0$, then $(g + g^{-1})z = 0$. Since $z^2 = 1$, we get that $g + g^{-1} = 0$. Also, we get that algebra $\mathcal{A}_{\pm i}$ has the required basis and the relations. Existence of canonical isomorphism is trivial.

For studying algebra \mathcal{A}_θ , we have to study center of algebra \mathcal{A}_θ . If $b \neq 0$ then this center has basis $1, g + g^{-1}$ with relation $(g + g^{-1})^2 = b^2 \cdot 1$. Further, center has two characters $g + g^{-1} \mapsto b = \theta + \theta^{-1}$ and $g + g^{-1} \mapsto -b = -(\theta + \theta^{-1})$. If $\theta = \pm i$, then center has a basis $1, z$ and relation $z^2 = 1$. Thus, there are two characters of center. Analogous to proposition 7, one can show that if $\theta \neq \pm 1$ algebra \mathcal{A}_θ is a direct sum of matrix algebras.

Using canonical isomorphism \mathcal{A}_θ and $\mathcal{A}_{-\theta}$, we get that it is sufficient to consider the case of algebra \mathcal{A}_1 . Using proposition 7, we get that if $\theta = 1$, algebras \mathcal{A}_1 has 4-dimensional Jacobson radical. Actually, in this case $(g^2 - 1)^2 = 0$. Consider ideal J of \mathcal{A}_1 generated by element $g^2 - 1$. Thus, ideal J has a basis $(g^2 - 1), x(g^2 - 1), g(g^2 - 1), xg(g^2 - 1)$. One can check that $J^2 = 0$. Therefore, algebras $\mathcal{A}_{\pm 1}$ has 4-dimensional Jacobson radical. \square

Further, let us consider image of algebras \mathcal{A}_θ under representation ϕ defined by matrices X, Y, Z . Remind the following notion: representation of algebra is called *semisimple* iff this representation is a direct sum of irreducible ones. Recall the following property of Jacobson radical: image of radical under any semisimple representation is zero. Subalgebra $\mathcal{L}_\theta \subset \text{Mat}_4(\mathbb{C})$ is an image of algebra \mathcal{A}_θ for arbitrary θ . Using canonical isomorphism $\mathcal{A}_\theta \cong \mathcal{A}_{-\theta}$, we get that b is defined up to sign. One can find that we can choose $b = \chi(z(g + g^{-1})) = \pm(\theta + \theta^{-1})$. Therefore, we have the following

Corollary 18. *Consider subalgebra $\mathcal{M}_\theta \subset \text{Mat}_4(\mathbb{C})$ generated by $\mathcal{L}_\theta \subset \text{Mat}_4(\mathbb{C})$. We have the following possibilities:*

- If $\theta \neq \pm 1$, then this \mathcal{M}_θ is a direct sum $\text{Mat}_2(\mathbb{C}) \oplus \text{Mat}_2(\mathbb{C})$.
- Assume that $\theta = \pm 1$. In this case subalgebra \mathcal{M}_θ is an image of Klein group or direct sum of four \mathbb{C} 's.

Appendix A. Homological properties of irreducible $\mathbb{C}G$ -modules.

In this section we will introduce some notions of homological algebra and prove some technical facts on some $\mathbb{C}G$ -modules. One can find classical definitions and results in homological algebra in many books, for example [13], [14], [12]

Recall the notion of extension group. Fix some associative algebra A . Suppose that we have two A -modules W_1 and W_2 . We would like to classify A -modules W satisfying the following conditions: W_1 is a submodule of W and quotient W/W_1 is isomorphic to W_2 . For this purpose we will introduce the following equivalence relation: we will say that W is equivalent to W' iff there is a following commutative diagram of A -modules:

$$\begin{array}{ccccccc} 0 & \longrightarrow & W_1 & \longrightarrow & W & \longrightarrow & W_2 \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & W_1 & \longrightarrow & W' & \longrightarrow & W_2 \longrightarrow 0 \end{array} \quad (41)$$

where all morphisms are isomorphisms. Standard arguments from homological algebra tell us that set of equivalence classes is an abelian group. This group is called extension group $\text{Ext}_A^1(W_2, W_1)$. Neutral element of this group is a direct sum $W_1 \oplus W_2$.

Extension group has the following description in the case of commutative algebras. Assume that A is a finite generated commutative algebra. Consider algebraic variety $\text{Spec} A =$

$\text{Hom}_{alg}(A, \mathbb{C})$. Let χ be a point of $\text{Spec}A$. \mathbb{C}^χ is a corresponding A -module. Then $\text{Ext}_A^1(\mathbb{C}^\chi, \mathbb{C}^\chi)$ is a tangent space of $\text{Spec}A$ at the point χ . Recall the following way of calculation of extension group $\text{Ext}_A^1(V_1, V_2)$ of A -modules V_1 and V_2 for arbitrary algebra A . Assume that we have a projective resolution of A -module V_1 :

$$0 \longrightarrow P_s \longrightarrow \dots \longrightarrow P_1 \longrightarrow V_1 \longrightarrow 0, \quad (42)$$

where $P_i, i = 1, \dots, s$ are projective A -modules. Applying to this resolution functor $\text{Hom}_A(-, V_2)$, we get the following complex (not exact sequence!):

$$0 \longrightarrow \text{Hom}_A(P_1, V_2) \longrightarrow \dots \longrightarrow \text{Hom}_A(P_s, V_2) \longrightarrow 0. \quad (43)$$

Zeroth cohomology group of this complex is a group $\text{Hom}_A(V_1, V_2)$, first cohomology group is a group $\text{Ext}_A^1(V_1, V_2)$. Note that there is a notion of extension group $\text{Ext}_A^i(V_1, V_2)$ of A -modules V_1 and V_2 . In this case $\text{Ext}_A^i(V_1, V_2)$ is a i th cohomology group of the complex (43).

Consider one-dimensional $\mathbb{C}P$ -module \mathbb{C}^χ , where χ is a character of $\mathbb{C}[g^{\pm 1}] \otimes \mathbb{C}[z, z^2 = 1]$. Restrict $\mathbb{C}P$ -module \mathbb{C}^χ to algebra of Laurent polynomials $\mathbb{C}[g^{\pm 1}]$. It can be shown in usual way that there is a free resolution (and, hence, projective resolution) of \mathbb{C}^χ as $\mathbb{C}[g^{\pm 1}]$ -module:

$$0 \longrightarrow \mathbb{C}[g^{\pm 1}] \xrightarrow{j} \mathbb{C}[g^{\pm 1}] \xrightarrow{p} \mathbb{C}^\chi \longrightarrow 0. \quad (44)$$

Actually, \mathbb{C}^χ is a quotient of $\mathbb{C}[g^{\pm 1}]$ by ideal generated by $g - \chi(g)1$. Morphism p is a natural projection: $p : \mathbb{C}[g^{\pm 1}] \rightarrow \mathbb{C}[g^{\pm 1}] / \langle g - \chi(g)1 \rangle$. Morphism j is a morphism $\mathbb{C}[g^{\pm 1}] \rightarrow \mathbb{C}[g^{\pm 1}]$ is generated by rule $j(1) = (g - \chi(g) \cdot 1)$ and, hence, $j(g) = g(g - \chi(g) \cdot 1)$. One can prove that this morphism is injective. Also, image $j(\mathbb{C}[g^{\pm 1}])$ coincides with $\text{Ker} p$. Further, come back to $\mathbb{C}P$ -module \mathbb{C}^χ . In this case \mathbb{C}^χ is a quotient of $\mathbb{C}P$ by ideal generated by elements $g - \chi(g)1, z - \chi(z)1$. It follows that $(z + 1)^2 = 2(z + 1)$ and $(1 - z)^2 = 2(1 - z)$. We have the following decomposition $\mathbb{C}P$ into direct sum of projective $\mathbb{C}P$ -modules: $\mathbb{C}P = \mathbb{C}P(z + 1) \oplus \mathbb{C}P(z - 1)$. It can be shown that there is a projective resolution of \mathbb{C}^χ as $\mathbb{C}P$ -module:

$$0 \longrightarrow \mathbb{C}P(z \pm 1) \xrightarrow{j} \mathbb{C}P(z \pm 1) \xrightarrow{p} \mathbb{C}^\chi \longrightarrow 0, \quad (45)$$

where we take plus or minus simultaneously in both modules as follows: if $\chi(z) = 1$ then we take plus, else we take minus.

Using standard arguments and $\mathbb{C}P$ -resolution of \mathbb{C}^χ , one can formulate the following proposition:

Proposition 19. *Consider two points $\chi, \psi \in \text{Spec} \mathbb{C}P$. Then we have the following statements:*

- If $\chi \neq \psi$, then $\text{Ext}_{\mathbb{C}P}^1(\mathbb{C}^\chi, \mathbb{C}^\psi) = 0$
- $\text{Ext}_{\mathbb{C}P}^1(\mathbb{C}^\chi, \mathbb{C}^\chi) = \mathbb{C}$.

Proof. Applying to resolution (45) functor $\text{Hom}_{\mathbb{C}[g, g^{-1}]}(-, \mathbb{C}^\psi)$, we get the following complex:

$$\text{Hom}_{\mathbb{C}P}(\mathbb{C}P(z \pm 1), \mathbb{C}^\psi) \rightarrow \text{Hom}_{\mathbb{C}P}(\mathbb{C}P(z \pm 1), \mathbb{C}^\psi).$$

One can show that if $\chi \neq \psi$ then this map is isomorphism, and hence, zeroth and first cohomology groups are trivial. In the case $\chi = \psi$ we obtain that $\text{Hom}_{\mathbb{C}P}(\mathbb{C}^\chi, \mathbb{C}^\chi) = \text{Ext}_{\mathbb{C}P}^1(\mathbb{C}^\chi, \mathbb{C}^\chi) = \mathbb{C}$. \square

Also, we can make some remarks on homological properties of so-called induced $\mathbb{C}G$ -modules.

Remark. Using resolution (45), we can get resolution of some $\mathbb{C}G$ -modules. Tensoring sequence (45) by $\mathbb{C}G$ over $\mathbb{C}P$, we get the following sequence:

$$0 \longrightarrow \mathbb{C}G \otimes_{\mathbb{C}P} \mathbb{C}P(z \pm 1) \longrightarrow \mathbb{C}G \otimes_{\mathbb{C}P} \mathbb{C}P(z \pm 1) \longrightarrow \mathbb{C}G \otimes_{\mathbb{C}P} \mathbb{C}^\chi \longrightarrow 0. \quad (46)$$

Module $\mathbb{C}G \otimes_{\mathbb{C}P} \mathbb{C}^\chi$ is called module *induced* by character χ of subgroup P . Sequence (46) is a projective resolution of $\mathbb{C}G \otimes_{\mathbb{C}P} \mathbb{C}^\chi$, i.e. $\mathbb{C}G \otimes_{\mathbb{C}P} \mathbb{C}P(z \pm 1)$ is a projective $\mathbb{C}G$ - module.

Applying standard arguments to $\mathbb{C}G$ -modules $W_\chi = \mathbb{C}G \otimes_{\mathbb{C}P} \mathbb{C}^\chi$, $W_\psi = \mathbb{C}G \otimes_{\mathbb{C}P} \mathbb{C}^\psi$ and resolution (46). We get the following proposition:

Proposition 20. *We have the following isomorphisms for W_χ and W_ψ :*

- if $\chi \neq \psi, \psi^{-1}$, then
$$\text{Ext}_{\mathbb{C}G}^1(W_\chi, W_\psi) = 0. \quad (47)$$

- if $\chi = \psi, \psi^{-1} \neq \pm 1$

$$\text{Ext}_{\mathbb{C}G}^1(W_\chi, W_\psi) = \mathbb{C}, \quad (48)$$

- if $\chi = \psi = \pm 1$, then
$$\text{Ext}_{\mathbb{C}G}^1(W_\chi, W_\psi) = \mathbb{C}^2, \quad (49)$$

Proof. we have the following complex:

$$\text{Hom}_{\mathbb{C}G}(\mathbb{C}G \otimes_{\mathbb{C}P} \mathbb{C}P(z \pm 1), W_\psi) \longrightarrow \text{Hom}_{\mathbb{C}G}(\mathbb{C}G \otimes_{\mathbb{C}P} \mathbb{C}P(z \pm 1), W_\psi). \quad (50)$$

Zeroth and first cohomology group of this complex are isomorphic, i.e. $\text{Hom}_{\mathbb{C}G}(W_\chi, W_\psi) \cong \text{Ext}_{\mathbb{C}G}^1(W_\chi, W_\psi)$. Using adjacency of functors and corollary 11, we get that

$$\text{Hom}_{\mathbb{C}G}(W_\chi, W_\psi) \cong \text{Hom}_{\mathbb{C}P}(\mathbb{C}^\chi, \mathbb{C}^\psi \oplus \mathbb{C}^{c_x(\psi)}).$$

Using this isomorphism, we obtain the statement of the proposition. \square

Appendix B. Some aspects of noncommutative geometry in representation theory of G .

We can describe the connection between $\mathbb{C}G$ - modules and $\mathbb{C}P$ - modules from point of view of noncommutative algebraic geometry also. One can find many aspects of noncommutative algebraic geometry in many papers, for example, [11], [15], [16].

Consider a finite-generated associative algebra A . Recall that a *variety of representations* of algebra A is called the variety $\mathbf{Rep}_n(A) = \text{Hom}_{alg}(A, \text{Mat}_n(\mathbb{C}))$. Also, there is a natural action of $\text{GL}_n(\mathbb{C})$ on $\text{Mat}_n(\mathbb{C})$ by conjugation. Equivalently, if we fix basis in n -dimensional space \mathbb{C}^n , then we have an isomorphism: $\text{Mat}_n(\mathbb{C}) \cong \text{End}_{\mathbb{C}}(\mathbb{C}^n)$. In this case, the group $\text{GL}_n(\mathbb{C})$ acts on $\text{Mat}_n(\mathbb{C})$ by substitutions of bases in \mathbb{C}^n . Using this action, we have well-defined action of $\text{GL}_n(\mathbb{C})$ on $\mathbf{Rep}_n(A)$. Thus, we can consider the quotient of $\mathbf{Rep}_n(A)$ by action of $\text{GL}_n(\mathbb{C})$. This quotient is called by a *moduli variety* of algebra A . We will denote this variety by $\mathcal{M}_n(A)$ for an arbitrary algebra A .

We can consider any element $a \in A$ as a matrix-valued function on $\mathbf{Rep}_n(A)$. Namely, $a(\rho) = \rho(a)$, $\rho \in \mathbf{Rep}_n(A)$. Also, we can define $\text{GL}_n(\mathbb{C})$ - invariant functions on $\mathbf{Rep}_n(A)$ in the following manner: $\text{Tr}(a)(\rho) = \text{Tr} \rho(a)$, $\rho \in \mathbf{Rep}_n(A)$. Using $\text{GL}_n(\mathbb{C})$ - invariance, we can consider these functions as functions on $\mathcal{M}_n(A)$.

Recall the following well-known result.

Proposition 21. [11] *For a finite-generated algebra A , the ring of regular polynomial functions on $\mathcal{M}_n(A)$ is generated by functions Tra , $a \in A$.*

We will say that A -module V has *Jordan-Holder composite factors* (briefly, composite factors) W_1, \dots, W_s if there is a sequence of A -submodules of the following type:

$$0 = M_0 \subset M_1 \subset M_2 \dots \subset M_{s-1} \subset M_s = V \quad (51)$$

and the quotients $W_i = M_i/M_{i-1}$, $i = 1, \dots, s$ are irreducible A -modules. It is known that the set of composite factors for any A -module is unique up to permutation. Let us denote $\mathbf{gr}(V)$ the direct sum $W_1 \oplus \dots \oplus W_s$ for A -module V with composite factors W_1, \dots, W_s . Two n -dimensional A -modules V_1 and V_2 correspond to the same point of $\mathcal{M}_n(A)$ iff $\mathbf{gr}(V_1) \cong \mathbf{gr}(V_2)$. Equivalently, V_1 and V_2 correspond to the same point of $\mathcal{M}_n(A)$ iff traces of any elements $a \in A$ on V_1 and V_2 are the same, i.e. $\text{Tra}|_{V_1} = \text{Tra}|_{V_2}$ for any element $a \in A$.

Let us come back to our situation. We have an immersion of algebras: $\mathbb{C}P \rightarrow \mathbb{C}G$. Thus, we have induced maps:

$$p_1 : \mathbf{Rep}_2(\mathbb{C}G) \rightarrow \mathbf{Rep}_2(\mathbb{C}P) \quad (52)$$

and

$$p_2 : \mathcal{M}_2(\mathbb{C}G) \rightarrow \mathcal{M}_2(\mathbb{C}P). \quad (53)$$

Let us describe variety $\mathbf{Rep}_2 \mathbb{C}P$ as follows. Up to $\text{GL}_2(\mathbb{C})$ - conjugacy, we have three possibilities for picking up matrix corresponding to element g :

$$1. \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}, 2. \begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix}, 3. \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}, \alpha, \beta \in \mathbb{C}^*. \quad (54)$$

Recall that z commutes with g . Thus, we have the following possibilities for z in the first and third cases:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (55)$$

In the second case, we have the following situation for element z :

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (56)$$

Further, consider $\mathcal{M}_2(\mathbb{CP})$. It can be shown that the ring of regular polynomial functions on $\mathcal{M}_2(\mathbb{CP})$ is generated by $\text{Tr}(g)$, $\text{Tr}(g^2)$, $\text{Tr}(z)$. Also, function $\text{Tr}(z)$ can have only three possible values $\{-2, 0, 2\}$. It means that every point of $\mathcal{M}_2(\mathbb{CP})$ is defined by two continuous parameters $\text{Tr}(g)$, $\text{Tr}(g^2)$ and discrete parameter $\text{Tr}(z)$. We have the following proposition:

Proposition 22. *Variety $\mathcal{M}_2(\mathbb{CP})$ has three irreducible components: U_- , U_0 and U_+ . Every component is Zariski - open subset of affine plane \mathbb{C}^2 given by relation: $(\text{Tr}g)^2 - \text{Tr}g^2 \neq 0$. These components are indexed by values $\text{Tr}(z) = -2, 0, 2$ respectively.*

Proof. Recall that we have the following relation for 2×2 matrices:

$$\det(g) = \frac{1}{2}((\text{Tr}g)^2 - \text{Tr}g^2). \quad (57)$$

Since g is invertible, $\det(g) \neq 0$. Thus, we get the required statement. \square

Consider variety $\mathbf{Rep}_2\mathbb{C}G$. We have the following cases for matrices corresponding to x, y, z :

- x, y, z are reflections,
- z is scalar,

Consider first case. In this case x, y and z are reflections. Recall that z commutes with x and y . Note the following proposition:

Proposition 23. *If two reflections x and z commute, then there is a basis in which x and z are diagonal matrices.*

Proof. Pick up eigenvectors v_1, v_2 of z such that $zv_1 = v_1, zv_2 = -v_2$. We have the following relation: $xzv_1 = xv_1$ and $xzv_1 = xv_1$. Hence, xv_1 is eigenvector corresponding to eigenvalue 1. Therefore, $xv_1 = \alpha v_1$ for some $\alpha \in \mathbb{C}^*$. Since x is a reflection, then $\alpha = \pm 1$. The rest is analogous. \square

Thus, if x, y, z are reflections, then x, y, z commuting operators. Thus, x, y, z have the following view:

$$\begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix} \quad (58)$$

Consider second case. If x and y correspond to commuting matrices, then this case is quite similar to first case. If x and y are not commuting then matrix of z is scalar.

Proposition 24. *We have the following decomposition:*

$$\mathcal{M}_2(\mathbb{C}G) = \mathbb{C}_+^1 \cup \mathbb{C}_-^1 \cup S, \quad (59)$$

where $\mathbb{C}_+^1, \mathbb{C}_-^1$ are 1-dimensional families of $\mathbb{C}G$ - modules corresponding to $\text{Tr}z = \pm 2\text{Tr}x = \text{Tr}y = 0$ respectively, S is a set of isolated points, $|S| = 27$.

Proof. Using proposition 21, we get that the ring of regular polynomial functions on $\mathcal{M}_2(\mathbb{C}G)$ has the following generators: $\text{Tr}(x), \text{Tr}(y), \text{Tr}(z), \text{Tr}(xy), \text{Tr}(xz), \text{Tr}(yz)$. We have the following cases:

- x, y, z are reflections and correspond to matrices of type (58). In this case we have $\text{Tr}z = \text{Tr}x = \text{Tr}y = 0$ and $(\text{Tr}xy, \text{Tr}xz, \text{Tr}yz) = (2, -2, -2), (-2, 2, -2), (-2, -2, 2)$. In this case there are 3 points in S .
- y, z are reflections, x is a scalar matrix. In this case y and z are matrices of type (58). Thus, $\text{Tr}z = \text{Tr}y = 0$, $\text{Tr}x = \pm 2$, $\text{Tr}xy = \text{Tr}xz = 0$ and $\text{Tr}yz = \pm 2$. In this case there are 2 points in S .
- x, z are reflections, y is a scalar matrix. This case is analogous to second one. Thus, $\text{Tr}z = \text{Tr}x = 0$, $\text{Tr}y = \pm 2$, $\text{Tr}xy = \text{Tr}yz = 0$ and $\text{Tr}xz = \pm 2$. In this case there are 2 points in S .
- x is a reflection, y, z are scalar matrices. In this case, $\text{Tr}x = 0$, $\text{Tr}y = \pm 2$, $\text{Tr}z = \pm 2$ and $(\text{Tr}xy, \text{Tr}xz, \text{Tr}yz) = (0, 0, 2), (0, 0, -2)$. In this case there are 4 points in S .
- y is a reflection, x, z are scalar matrices. In this case, $\text{Tr}y = 0$, $\text{Tr}x = \pm 2$, $\text{Tr}z = \pm 2$ and $(\text{Tr}xy, \text{Tr}xz, \text{Tr}yz) = (0, 2, 0), (0, -2, 0)$. In this case there are 4 points in S .
- x, y, z are scalar matrices. In this case $\text{Tr}x = \pm 2, \text{Tr}y = \pm 2, \text{Tr}z = \pm 2$, $(\text{Tr}xy, \text{Tr}xz, \text{Tr}yz) = (2, 2, 2), (2, -2, -2), (-2, 2, -2), (-2, -2, 2)$. In this case there are 8 points in S .
- z is reflection, x, y are scalar matrices. In this case $\text{Tr}z = 0$, $\text{Tr}x = \pm 2$, $\text{Tr}y = \pm 2$ and $\text{Tr}xy = \pm 2, \text{Tr}xz = 0, \text{Tr}yz = 0$. In this case there are 4 points in S .
- z is scalar and x, y are reflections. In this case $\text{Tr}z = \pm 2$ and $\text{Tr}x = \text{Tr}y = 0$. Also, $\text{Tr}xz = \text{Tr}yz = 0$. In this case $\mathbb{C}G$ - modules are parameterized by $\text{Tr}xy$. In this case we have two 1-dimensional families $\mathbb{C}G$ -modules. We will denote two components corresponding to $\text{Tr}z = 2, \text{Tr}x = \text{Tr}y = 0$ and $\text{Tr}z = -2, \text{Tr}x = \text{Tr}y = 0$ by \mathbb{C}_+^1 and \mathbb{C}_-^1 respectively.

□

Let us come back to map p_2 .

Proposition 25. *Varieties $p_2(\mathbb{C}_-^1)$ and $p_2(\mathbb{C}_+^1)$ are curves given by equations:*

$$\text{Tr}^2(g) - \text{Tr}(g^2) = 2. \quad (60)$$

in U_- and U_+ respectively.

Proof. One can check that $p_2(\mathbb{C}_-^1) \subset U_-$ and $p_2(\mathbb{C}_+^1) \subset U_+$. Also, remind that $g = xy$. Thus, in the 1-dimensional components, we have the following identity: $\det g^2 = \det xyxy = \det^2 x \cdot \det^2 y$. Since x and y are reflections; we get that $\det g^2 = 1$. Using Gamilton - Cayley theorem, we have the following relation for g :

$$g^2 - \text{Tr}(g) \cdot g + \det g \cdot 1 = 0. \quad (61)$$

Since $\det g = 1$, we get that $g + g^{-1} = \text{Tr}(g) \cdot 1$ and $\text{Tr}(g) = \text{Tr}(g^{-1})$. Also, we have the relation:

$$\frac{\text{Tr}^2(g) - \text{Tr}(g^2)}{2} = \det g = 1. \quad (62)$$

We get that $p_2(\mathbb{C}_-^1)$ and $p_2(\mathbb{C}_+^1)$ are curves given by (62) in U_- and U_+ respectively. \square

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